Assignment-4

Ans1) Define groups. Give an example of Abelian group, and also of a Non-Abelian group.

Soln: group is a set of elements which fulfill properties as given below-:

(i) closure law
(ii) Associative law
(iii) Identity law
(iv) Inverse law

Eg-of Abelian group-:

$(Z_n, +_n)$ always shows abelian group property because it fulfill all 5 laws i.e closure law, Associative law, Identity law, Inverse law of commutative law.

Eg. of Non Abelian group-:

$(Z_n, \times_n)$ is Not an abelian group because it does Not show inverse property. Thus $(Z_n, \times_n)$ is abelian monoid.

Ques2) Let G be the set of integer Z. Define an operation * on G as
$$a*b = a+b+1 \text{ for } a,b \in Z$$
Justify that $(Z,*)$ is an abelian group.

Soln: (i) closure law-: because $a,b \in Z$ then sum of these element should be also integer that's why $a*b \in Z$

(ii) Associative law-: let $c \in Z$
$$(a*b)*c = (a+b+1)*c$$
$$= (a+b+c+2)$$
$$= (a+b+1+b+1)$$
$$= (a*c)*b$$

$$\boxed{(a*b)*c = (a*c)*b}$$

(iii) Existance OF Identity-: let e be identity

$\forall a \in G$ we have $a*e = a$
$$a+e+1 = a$$
$$\boxed{e = -1}$$

(iv) Inverse-: $a*b = e$
$$a+b+1 = -1$$
$$\boxed{b = -2-a}$$

(v) commutative-:

$$a*b = a+b+1$$

$$\text{also } b*a = a+b+1$$

$$\boxed{a*b = b*a}$$

Ques3 Let $G = Q\{-1\}$ and define an operations on G as $a*b = a+b+ab$ for $a, b \in G$ justify that, wrt operation $*$.

(i) closure-:

$$a*b = a+b-ab$$

$$a, b \in Q \text{ so } a*b \in Q$$

(ii) Associative-:

$$(a*b)*c = (a+b-ab)*c$$

$$= (a+b-ab)+c-(a+b-ab)c$$

$$= a+b+c-ab-ac-bc+abc$$

$$= a*(b+c-bc)$$

$$= a+(b+c-bc)-a(b+c-bc)$$

$$= a+b+c-bc-ac-ab+abc$$

(iii) Identity-:

Let e be identity

$\forall\ a \in Q$ we have $a*e = a$

$$a+e-ae = a$$

$$\boxed{e=0} \quad \boxed{a \neq 1}$$

(iv) Inverse-: Let b is the inverse of a.

$$a * b = e$$

$$a + b - ab = 0$$

$$\boxed{b = \frac{-a}{1-a}} \text{ or } \boxed{b = \frac{a}{a-1} \in Q}$$

(v) commutative-:

$$a * b = a + b + ab$$

also,

$$b * a = b + a + ba$$

$$\boxed{a * b = b * a}$$

given group G accomplish all 5 rules to given group G is an abelian group.

Ques4) $G = Q^* \times Q$ and define an operation $*$ on G as $(a,b) * (x,y) = (ax, ay+b)$ for $a,b,x,y \in Q$

Soln (i) closure-: $(a,b) * (x,y) = (ax, ay+b)$

for $a, b, x, y \in Q$

so, then,

also $(a,b) * (x,y) \in Q$

(ii) Associative-: Let $(e,f) \in Q$

$$((a,b) * (x,y)) * (e,f)$$

$$= (ax, ay+b) * e$$

$$= (axe, axf + (ay+b))$$

$$((a,b)*(c,f))*(x,y)$$
$$=(ac,\ af+b)*(x,y)$$
$$=(a\times c,\ (a\times f+b)+ay)$$

$$\boxed{((a,b)*(x,y))*(c,f) = ((a,b)*(c,f))*(x,y)}$$

(iii) Identity:

Let $\forall\ (a,b)\in G\ \exists$ an element $(1,0)\in G$

$\begin{pmatrix} a\neq 0 \\ a,b\in R \end{pmatrix}$ $\begin{pmatrix} 1\neq 0 \\ 1,0\in R \end{pmatrix}$

If $(a,b)*(c,d)=(1,0)$

$(ac,\ bc+d)=(1,0)$

$\boxed{ac=1}$ $\boxed{bc+d=0}$

$\boxed{c=1/a}$, $\boxed{d=-bc=-b/a}$

$\therefore \left(\frac{1}{a}, \frac{b}{a}\right)$ is inverse

(v) Commutative-:

$(a,b)*(x,y)=(ax,\ ay+b)$

$(x,y)*(a,b)=(xa,\ xb+y)$

Does Not satisfies commutative Property
Hence it's Not an abelian group.

Ques Let $(G, *)$ be a group. Prove that for all $a, b \in G$ we have

(i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

(i) Let e be the identity

$$a^{-1}a = e \quad \forall a \in G$$
$$(a^{-1})^{-1}(a^{-1}a) = (a^{-1})^{-1} e$$
$$((a^{-1})^{-1}a^{-1})a = (a^{-1})^{-1}$$
$$ea = (a^{-1})^{-1}$$
$$\boxed{a = (a^{-1})^{-1}}$$

(ii) Let a & b are any element of a group G and a^{-1} & b^{-1} are resp. the inverse of a & b then.

$$a \cdot a^{-1} = e = a^{-1}a \quad \text{--- (1)}$$
$$bb^{-1} = e = b^{-1}.b \quad \text{--- (2)}$$

$$\text{Now } (ab)(b^{-1}a^{-1}) = ((ab)b^{-1})a^{-1}$$
$$= (a(bb^{-1}))a^{-1}$$
$$= (ae)a^{-1}$$
$$= aa^{-1}$$
$$= e$$

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$\therefore \boxed{(ab)(b^{-1}a^{-1}) = e}$ — (3)

Similarly, $b^{-1}a^{-1}(ab) = b^{-1}(a^{-1}(ab))$

$= b^{-1}((a^{-1}a)b)$ (Ass. Law)

$= b^{-1}(eb)$ From-1

$= e$ From-2

$\boxed{(b^{-1}a^{-1})(ab) = e}$ — (4)

Hence $(ab)^{-1} = b^{-1}a^{-1}$

$\forall\ a,b \in G$

__Ques__ Justify that a group G is abelian if & only if $(ab)^2 = a^2b^2$ for all $a,b \in G$

__Soln:__ a, b are arbitrary elements of a group then $(ab)^2 = (a^2b^2)$ iff G is abelian.

$\Rightarrow (ab)^2 = a^2b^2$

$\Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow \cancel{a}(ba)\cancel{b} = \cancel{a}(ab)\cancel{b}$

$\therefore \boxed{ab = ba}\ \forall \boxed{a,b \in G}$

Hence G is abelian group.

conversely, suppose that,

$$(ab)^2 = a^2b^2 \quad \forall\ a,b \in G$$

$$(ab)^2 = (ab)(ab)$$
$$= a(ba)b$$
$$= a(ab)b$$
$$= (aa)(bb)$$

$$\boxed{(ab)^2 = a^2b^2 \quad \forall\ a,b \in G}$$

Ques Define cyclic group. Justify that every cyclic group is abelian.

Soln. Cyclic group -: Let G be any group. If for $a \in G$ every element $x \in G$ can be generated by a such that $x = a^n$, $n \in I$. Then G is called a cyclic group and a is called its generator.

A cyclic group G with generator a is denoted by.

$$G = \langle a \rangle \text{ or } G = \{a\}$$

Proof-:

Let $G = \langle a \rangle$ be a cyclic group generated by a then to show that G is an abelian group let $x, y \in G$ then $\exists$ $r, s \in I$ such that.

$$x = a^r, \quad y = a^s$$

$$\text{Now } xy = a^r \cdot a^s$$
$$= a^{r+s}$$
$$= a^s \cdot a^r$$
$$= y \cdot x$$

$$\boxed{xy = yx} \quad \forall \boxed{x, y \in G}$$

Hence G is an abelian group.

Ques Justify that every cyclic group is abelian give eg of a Non cyclic abelian group of order 4.

Soln. Let $G = \langle a \rangle$ be a cyclic group generated by a then to show that G is abelian group.

Let $x, y \in G$ then $\Rightarrow \exists r, s \in I$ such that
$x = a^r$ & $y = a^s$

Now $xy = a^r \cdot a^s$
$= a^{r+s}$
$= a^{s+r}$
$= a^s \cdot a^r$
$= y \cdot x$

$\boxed{xy = yx} \quad \forall x, y \in G$

Hence G is an abelian group.

Example -: 4 group is example of a Non cyclic group of order 4

i.e $((\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}))$
$V = \{e, a, b, c\}$

Ques Justify that set $C_4 = \{1, -1, i, -i\}$ is a group with respect to multiplication of complex so. Find the order of elements of the group C_4. Is C_4 a cyclic group? If yes find all its generators?

Soln: Given group $C_4 = \{1, -1, i, -i\}$
so. order of each element in group.

(i) $O(1)$ {order of 1 in group}
$= 1$

(ii) $(-1)^2 = (-1)(-1) = 1$
$\boxed{O(-1) = 2}$

(iii) $(i)(i)(i)(i) = 1$
$i^4 = 1$
$\boxed{O(i) = 4}$

(iv) $(-i)(-i)(-i)(-i) = 1$
$(-i)^4 = 1$
$\boxed{O(-i) = 4}$

yes C_4 is cyclic group.
given $C_4 = \{1, -1, i, -i\}$
$C_4 = \{i^4, i^2, i, i^3\}$
$C_4 = \{i, i^2, i^3, i^4\}$

$\boxed{C_4 = \langle i \rangle}$ Hence C_4 is cyclic and i is generator of C_4.

and also we can write as,

$$C_4 = \{(-i)^4, (-i)^3, (-i)^2, -i\}$$

$$C_4 = \{-i, (-i)^2, (-i)^3, (-i)^4\}$$

Hence $-i$ is also generator of C_4

therefore $C_4 \langle i \rangle$ & $C_4 \langle -i \rangle$

Ques Justify intersection of any two subgroups of a group is also subgroup of the group. Give an example of two subgroups. Give an example of two subgroups of a group whose union is not a subgroup.

Soln: Let H_1 & H_2 are two subgroups of group G. Then show that $H_1 \cap H_2$ is a subgroup G.

Since $e \in H_1$, $e \in H_2$

$H_1 \cap H_2 \neq \phi$

Let $a, b \in H_1 \cap H_2$

$a, b \in H_1$ and $a, b \in H_2$

$ab^{-1} \in H_1$ and $ab^{-1} \in H_2$

$$\boxed{ab^{-1} \in H_1 \cap H_2}$$

Hence $H_1 \cap H_2$ is also a subgroup of group G.

Eg-:
let $H_1 = \{ \ldots -15, -10, -5, 0, 5, 10, 15 \ldots \}$
& $H_2 = \{ \ldots -6, -4, -2, 0, 2, 4, 6 \ldots \}$
are subgroups of $(Z, +)$

Then,
$H_1 \cup H_2 = \{ 0, \pm 2, \pm 4, \pm 6 \ldots ; 0, \pm 5, \pm 10, \pm 15 \ldots \}$
since $4 \in H_1 \cup H_2$, $5 \in H_1 \cup H_2$
$4 + 5 = 9 \notin H_1 \cup H_2$

$H_1 \cup H_2$ is Not closed.

Ques State and prove Lagrange's theorem.

Statement -: The order of each subgroup. Finite group is a divisor of the order of group.

Proof-: Let G is a Finite group of order n
i.e $\boxed{O(G) = n}$ $G = \{a_1, a_2 \ldots a_n\}$

and H is subgroup of G of order m.
i.e $\boxed{O(H) = m}$ $\therefore H = \{ h_1, h_2 \ldots h_m$

Then to show that

$$O(H) / O(G) \text{ or } \frac{O(G)}{O(H)} \text{ or } n/m$$

since $a_1 \in G$.

Then $Ha_1 = \{h_1a_1, h_2a_1 \ldots h_ma_1\}$ be right coset OF H in G having m distinct elements.

because $h_ia = h_ja$ $\{i \leq j, j \leq m\}$

$(h_i = h_j)$

$(O(H) \neq m)$

$(O(Ha_1) = m)$

Therefore, each right coset of H in G have m distinct elements.

Now, suppose there are K distinct cosets of H in G have m distinct elements

$Ha_1 = \{h_1a_1, h_2a_1 \ldots h_ma_1\}$

$Ha_2 = \{h_1a_2, h_2a_2 \ldots h_ma_2\}$

$\vdots$

$Ha_k = \{h_1a_k, h_2a_k \ldots h_ma_k\}$

know that the union of all left or right cosets of H is equal to the group i.e.

$$G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_k$$

$$O(G) = O(Ha_1) + O(Ha_2) + \cdots O(Ha_k)$$

$$n = m + m + \cdots m \ (k \text{ time})$$

$$n = mk$$

$$\boxed{k = \frac{n}{m}} = \frac{O(G)}{O(H)}$$

Hence $O(H)$ is divisor of $O(G)$

Ques $Z_6 = \{0, 1, 2, 3, 4, 5\}$

(i) Justify that $(Z_6, +_6)$ is a group.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

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(i) closed -: On operating operations on set Z_6, all elements lies under the set $Z_6 = \{0, 1, 2, 3, 4, 5\}$ so, it shows closed property

(ii) Associative law -:

$(3 +_6 4) +_6 5 = 1 +_6 5 = 0$
$(3 +_6 5) +_6 4 = 2 +_6 4 = 0$

(iii) Identity -: identity is zero for given set.

(iv) Inverse -:

$0' = 0 \quad , 4' = 2$
$1' = 5 \quad , 5' = 1$
$2' = 4$
$3' = 3$

$\therefore$ set Z_6 satisfies 4 rules i.e closure associative existence of identity of inverse Hence it is a group.

(ii) $(Z_6 | \{0\}, \times_6\}$

$\times_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	2	2
5	5	4	3	2	1

Not an abelian group bcz Not satisfies the identity, closure, inverse property.

Ques Find cosets of the subgroup $H = 5\mathbb{Z}$ in the group $(\mathbb{Z}, +)$ what is the index $(G:H)$?

Soln. $\mathbb{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \}$

given $H = 5\mathbb{Z}$

$H = \{ \ldots -15, -10, -5, 0, 5, 10, 15 \ldots$

$H+0 = \{ \ldots -15, -10, -5, 0, 5, 10, 15 \ldots \}$

$H+1 = \{ \ldots -14, -9, -4, 1, 6, 11, 16 \ldots \}$

$H+2 = \{ \ldots -13, -8, -3, 2, 7, 12, 17 \ldots \}$

$H+3 = \{ \ldots -12, -7, -2, 3, 8, 13, 18 \ldots \}$

$H+4 = \{ \ldots -11, -6, -1, 4, 9, 14, 19 \ldots \}$

$H+5 = \{ \ldots -10, -5, 0, 5, 10, 15, 20 \ldots \} = H$

Hence index $[G:H] = 5$